

Last time: More Double Integration

Q: What is the volume of a sphere of radius $a > 0$?

Sol: (w/ what we know)

picture



$$\text{Vol}(S_a) = \iint_{R_a} h(x,y) \, dA$$

$$x^2 + y^2 + z^2 = a^2$$

The sphere with the equation $x^2 + y^2 + z^2 = a^2$ has an upper hemisphere with equation $z = \sqrt{a^2 - x^2 - y^2}$ and the lower hemisphere w/ an equation $z = -\sqrt{a^2 - x^2 - y^2}$

"Height Function = (Upper hemisphere) - (Lower hemisphere)"

$$h(x,y) = \sqrt{a^2 - x^2 - y^2} - (-\sqrt{a^2 - x^2 - y^2}) = 2\sqrt{a^2 - x^2 - y^2}$$

Now, the region of integration is:

$$R = \{(x,y) : x^2 + y^2 \leq a^2\}$$

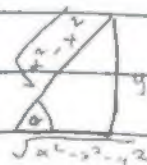
Now the upper semi-circle boundary R_a is $y = \sqrt{a^2 - x^2}$ and lower semi-circle is $y = -\sqrt{a^2 - x^2}$

$$R_a = \{(x,y) : -a \leq x \leq a, -\sqrt{a^2 - x^2} \leq y \leq \sqrt{a^2 - x^2}\}$$

$$\text{Hence, } \text{Vol}(S_a) = \int_{x=-a}^a \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2 - x^2 - y^2} \, dy \, dx$$

Inner Int

$$\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2 - x^2 - y^2} \, dy$$



$$= \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2 - x^2 - y^2} \, dy$$

$$= 2 \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2} \, dy$$

$$\sin(\theta) = \frac{y}{\sqrt{a^2-x^2}}$$

$$= (a^2 - x^2) \int_{-\cos \theta}^{\cos \theta} \cos^2 \theta \, d\theta$$

$$y = \sqrt{a^2 - x^2} (\sin \theta)$$

$$= (a^2 - x^2) \int_{-\cos \theta}^{\cos \theta} (1 + \cos(2\theta)) \, d\theta$$

$$dy = \sqrt{a^2 - x^2} (\cos \theta) \, d\theta$$

$$= (a^2 - x^2) \left(\theta + \frac{1}{2} \sin(2\theta) \right) + C$$

$$\sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - x^2} (\cos \theta)$$

$$= (a^2 - x^2) (\theta + \sin \theta \cos \theta) + C$$

$$= (\alpha^2 - x^2) \left(\arcsin\left(\frac{y}{\sqrt{\alpha^2 - x^2}}\right) + \frac{y}{\sqrt{\alpha^2 - x^2}} \cdot \frac{\sqrt{\alpha^2 - x^2 - y^2}}{\sqrt{\alpha^2 - x^2}} \right) + C$$

$$= (\alpha^2 - x^2) \arcsin\left(\frac{y}{\sqrt{\alpha^2 - x^2}}\right) + y \sqrt{\alpha^2 - x^2 - y^2} + C$$

$$= \int_{y=-\sqrt{\alpha^2-x^2}}^{\sqrt{\alpha^2-x^2}} 2 \sqrt{\alpha^2 - x^2 - y^2} dy$$

$$= \left[(\alpha^2 - x^2) \arcsin\left(\frac{y}{\sqrt{\alpha^2 - x^2}}\right) + y \sqrt{\alpha^2 - x^2 - y^2} \right]_{-\sqrt{\alpha^2-x^2}}^{\sqrt{\alpha^2-x^2}}$$

$$= \left((\alpha^2 - x^2) \arcsin(1) + \sqrt{\alpha^2 - x^2} \sqrt{0} \right) - \left((\alpha^2 - x^2) \arcsin(-1) - \sqrt{\alpha^2 - x^2} \sqrt{0} \right)$$

$$= (\alpha^2 - x^2) (\arcsin(1) - \arcsin(-1)) = (\alpha^2 - x^2) \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right) = \boxed{\pi(\alpha^2 - x^2)}$$

Outer Int

$$\int_{x=-\alpha}^{\alpha} \pi(\alpha^2 - x^2) dx$$

$$= \pi \left(\alpha^2 x - \frac{1}{3} x^3 \right) \Big|_{-\alpha}^{\alpha}$$

$$= \pi \left(\left(\alpha^3 - \frac{1}{3} \alpha^3 \right) - \left(-\alpha^3 + \frac{1}{3} \alpha^3 \right) \right)$$

$$= \pi \alpha^3 \left(2 - \frac{2}{3} \right) = \boxed{\frac{4}{3} \pi \alpha^3} \leftarrow \text{Vol}(S_\alpha)$$

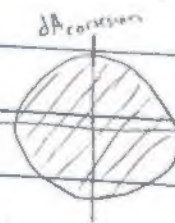
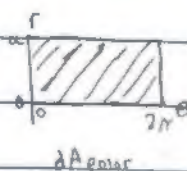
NB: This computation above was complicated. It seems it would be more natural to use polar coordinates to describe both the region and height function.

The region:



Similarly the height function:

$$h(r \cos \theta, r \sin \theta) = 2 \sqrt{\alpha^2 - r^2}$$



$$\begin{aligned} & \text{Cartesian} \\ & x^2 + y^2 \leq \alpha^2 \\ & 0 \leq r \leq \alpha \\ & 0 \leq \theta \leq 2\pi \end{aligned} \quad \text{Polar}$$

Need: How is the differential affected?

Want: Formula for dA_{cylinder} in terms of dA_{cyl}

(Think abt a circular sector)



From high school geometry: Area of full circle section is $\frac{1}{2}\theta \cdot r^2$

The circular sector corresponding to polar rectangle $[\theta_1, \theta_2] \times [r_1, r_2]$ has area $\frac{1}{2}(\theta_2 - \theta_1)r_2^2 - \frac{1}{2}(\theta_2 - \theta_1)r_1^2$
 $= \frac{1}{2}(\theta_2 - \theta_1)(r_2^2 - r_1^2)$
 $= \frac{1}{2}(r_1 + r_2)(\theta_2 - \theta_1)(r_2 - r_1)$

\therefore The corresponding areas are
 $\Delta A_{rect} = \frac{1}{2}(r_1 + r_2) \Delta\theta \Delta r = \frac{1}{2}(r_1 + r_2) \Delta A_{polar}$

Now limiting the $\Delta A_{polar} \rightarrow 0$ (via $\Delta\theta \rightarrow 0$ and $\Delta r \rightarrow 0$)
 we see that $\frac{1}{2}(r_1 + r_2) = \frac{1}{2}(r_2 - \Delta r) = r_2 + \frac{1}{2}\Delta r \rightarrow r$

So in the limit $dA_{rect} = r dA_{polar}$

Volume of sphere, take 2

Sol: Use Polar coordinates

$$\text{Vol}(S_a) = \iint_{R_{rect}} h(x, y) dA_{rect}$$

Go back & back at sphere?

$$z^2 = a^2 - r^2$$

$$= \iint_{R_{polar}} h(r \cos \theta, r \sin \theta) r dA_{polar}$$

$$z_{upper} = \sqrt{a^2 - r^2}$$

$$z_{lower} = -\sqrt{a^2 - r^2}$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^a 2\sqrt{a^2 - r^2} r dr d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^a -u^{1/2} du d\theta$$

Usub

$$= - \int_{\theta=0}^{2\pi} \left[\frac{2}{3} u^{3/2} \right]_0^a d\theta$$

$$= - \int_{\theta=0}^{2\pi} \left[\frac{2}{3} (a^2 - r^2)^{3/2} \right]_{r=0}^a d\theta$$

$$= -\frac{1}{3} \int_{\theta=0}^{2\pi} \left((x^2+y^2)^{3/2} - (x^2+y^2)^{3/2} \right) d\theta$$

$$= -\frac{2}{3} - x^3 \int_{\theta=0}^{2\pi} d\theta$$

$$= \frac{2}{3} x^3 [\theta]_{\theta=0}^{2\pi}$$

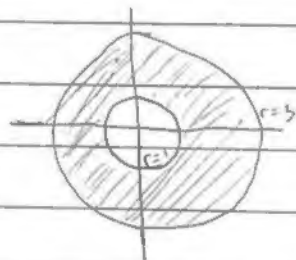
$$= \frac{4}{3\pi} x^3$$

Ex: Compute $\iint_R \cos(\sqrt{x^2+y^2}) dA$ for R the annulus between $x^2+y^2=1$ and $x^2+y^2=9$

Solution: Set this up using polar coordinates:

$$\cos(\sqrt{x^2+y^2}) = \cos(\sqrt{r^2}) = \cos(r) \quad (\text{b/c } r \geq 0)$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$



$$\therefore \iint_{R_{\text{annulus}}} \cos(\sqrt{x^2+y^2}) dA_{\text{cart}} = \iint_{R_{\text{polar}}} \cos(r) r dA_{\text{polar}}$$

$$R_{\text{polar}} = \{(r, \theta) : 1 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$$

$$= \int_{r=1}^3 \int_{\theta=0}^{2\pi} r \cos(r) d\theta dr$$

$$= \int_{r=1}^3 r \cos(r) [\theta]_{\theta=0}^{2\pi} dr$$

$$\begin{aligned} (u=r \quad dv = \cos(r) dr) \\ (du=dr \quad v = \sin(r)) \end{aligned}$$

$$= 2\pi \int_{r=1}^3 r \cos(r) dr$$

$$= 2\pi \left[r \sin(r) - \int \sin(r) dr \right]_{r=1}^3$$

$$= 2\pi \left[r \sin(r) + \cos(r) \right]_{r=1}^3$$

$$= 2\pi \left((3 \sin(3) + \cos(3)) - (\sin(1) + \cos(1)) \right)$$

Exercise: Compute $\iint_R y \exp(-x^2 - y^2) dA$ on the region R the quarter annulus in the first quadrant. Between $x^2 + y^2 = 25$ and $x^2 + y^2 = 4$.

